S.-T. Yau College Student Mathematics Contests 2025

Analysis and Differential Equations

Individual All-around

Please solve one out of the following two problems.

1. Let $f : \mathbb{R} \to \mathbb{R}$ be a measurable function supported on $[0,1]$. Let Mf denotes its maximal function:

$$\forall x \in \mathbb{R}, \quad Mf(x) = \sup_{r>0} \frac{1}{2r} \int_{x-r}^{x+r} |f(x)| \, \mathrm{d}x.$$

Show that if $\int_0^1 |f(x)| \max\{1, \log|f(x)|\} \, \mathrm{d}x < +\infty$, then

$$\int_0^1 Mf(x) \, \mathrm{d}x < +\infty.$$

2. Let $u(t,x) : \mathbb{R} \times \mathbb{R} \to \mathbb{C}$ be periodic in x variable, i.e. $\forall t, x$, one has $u(t, x + 2\pi) = u(t,x)$. Suppose $u(t,x)$ is smooth and solves the linear Schrödinger equation,

$$\begin{cases} i\partial_t u(t,x) + \partial_{xx} u(t,x) = 0, \\ u(0,x) = f(x). \end{cases} \tag{0.1}$$

Prove the following estimate

$$\int_0^{2\pi} \int_0^{2\pi} |u(t,x)|^4 \, \mathrm{d}x \, \mathrm{d}t \leq C\Big(\int_0^{2\pi} |f(x)|^2 \, \mathrm{d}x\Big)^2 \tag{0.2}$$

Here C is some universal constant.

Hint: Try to use Fourier series and Plancherel theorem. Also recall $\|F\|_{L^4}^2 = \|F^2\|_{L^2}$.